Boolean Function Optimization

- ✓ Minimizing the gate input (or literal) cost of a (a set of) Boolean equation(s) reduces circuit cost.
- ✓ We choose gate input cost.
- ✓ Boolean Algebra and graphical techniques are tools to minimize cost criteria values.
- ✓ Some important questions:
  - When do we stop trying to reduce the cost?
  - Do we know when we have a minimum cost?
- ✓ Treat optimum or near-optimum cost functions for two-level (SOP and POS) circuits first.
- ✓ Introduce a graphical technique using Karnaugh maps (K-maps, for short)

Karnaugh Maps (K-map)

- ✓ A K-map is a collection of squares
  - Each square represents a minterm
  - The collection of squares is a graphical representation of a Boolean function
  - Adjacent squares differ in the value of one variable
  - Alternative algebraic expressions for the same function are derived by recognizing patterns of squares (corresponding to cubes)
- ✓ The K-map can be viewed as

A reorganized version of the truth table or a particular cube representation

Some Uses of K-Maps

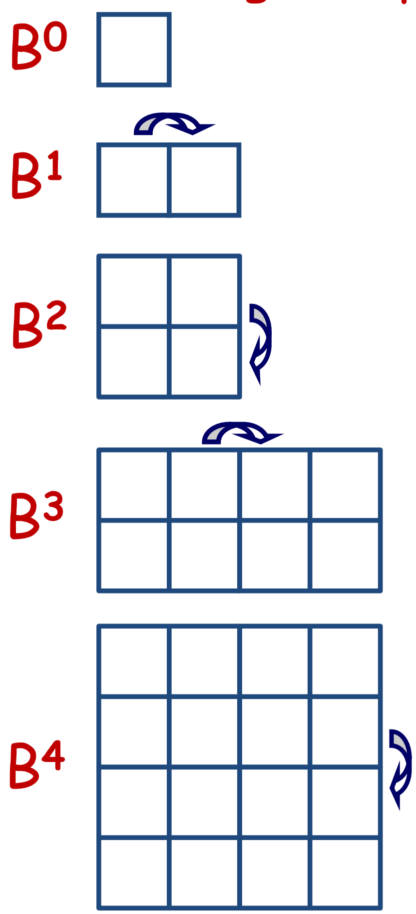
- ✓ Provide a means for:
  - Finding optimum
    - SOP and POS standard forms, and
    - two-level AND/OR and OR/AND circuit implementations

    for functions with small numbers of variables
  - Visualizing concepts related to manipulating Boolean expressions
  - Demonstrating concepts used by computer-aided design programs to simplify large circuits

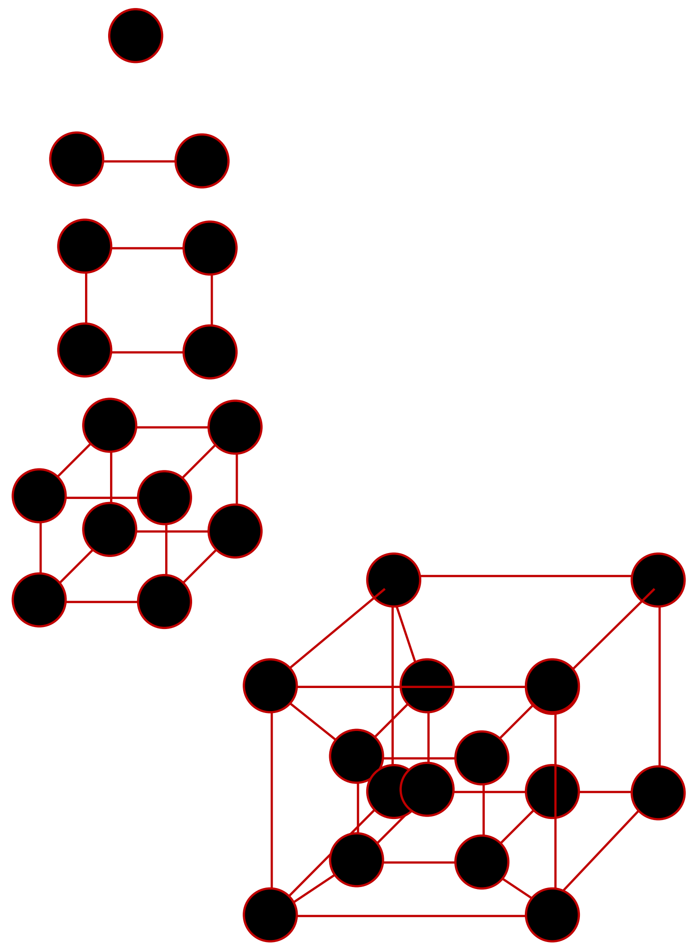
The Boolean Space B^n

- ✓ $B = \{0,1\}$
- ✓ $B^2 = \{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$

Karnaugh Maps:



Boolean Cubes:



Two Variable Maps

- ✓ A 2-variable Karnaugh Map:
  - Note that minterm m_0 and minterm m_1 are "adjacent" and differ in the value of the variable y
  - Similarly, minterm m_0 and minterm m_2 differ in the x variable.
  - Also, m_1 and m_3 differ in the x variable as well.
  - Finally, m_2 and m_3 differ in the value of the variable y

x \ y	y = 0	y = 1
x = 0	m_0 $\bar{x}\bar{y}$	m_1 $\bar{x}y$
x = 1	m_2 $x\bar{y}$	m_3 xy

K-Map and Truth Tables

- ✓ The K-Map is just a different form of the truth table.
- ✓ Example – Two variable function:
  - We choose a,b,c and d from the set {0,1} to implement a particular function, F(x,y).

Function Table

Input Values (x,y)	Function Value F(x,y)
0 0	a
0 1	b
1 0	c
1 1	d

K-Map

x \ y	y = 0	y = 1
x = 0	a	b
x = 1	c	d

K-Map Function Representation

- ✓ Example: F(x,y) = x

F = x	y = 0	y = 1
x = 0	0	0
x = 1	1	1

- ✓ For function F(x,y), the two adjacent cells containing 1's can be combined using the Minimization Theorem:

$$F(x,y) = x\overline{y} + xy = x$$

K-Map Function Representation

- ✓ Example: $G(x,y) = \overline{x}y + x\overline{y} + xy$

G=x+y	y = 0	y = 1
x = 0	0	1
x = 1	1	1

- ✓ For G(x,y), two pairs of adjacent cells containing 1's can be combined using the Minimization Theorem:

$$G(x,y) = (x\overline{y} + xy) + (\overline{x}y + xy) = x + y$$

Duplicate xy

Three Variable Maps

- A three-variable K-map:

x \ yz	yz=00	yz=01	yz=11	yz=10
x=0	m_0	m_1	m_3	m_2
x=1	m_4	m_5	m_7	m_6

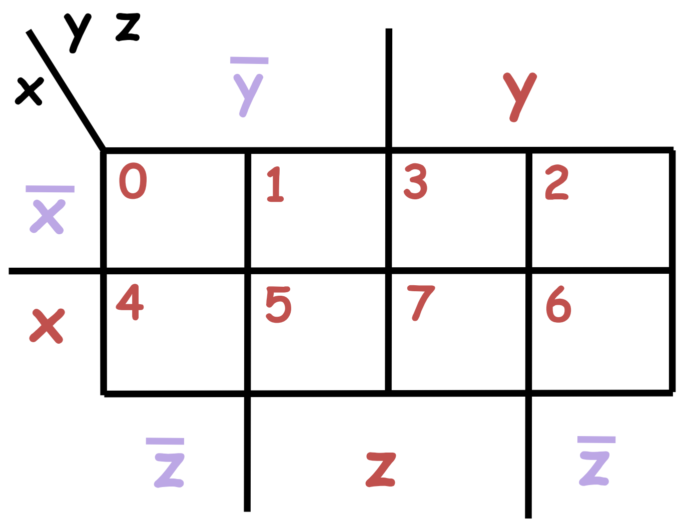
- Where each minterm corresponds to the product terms:

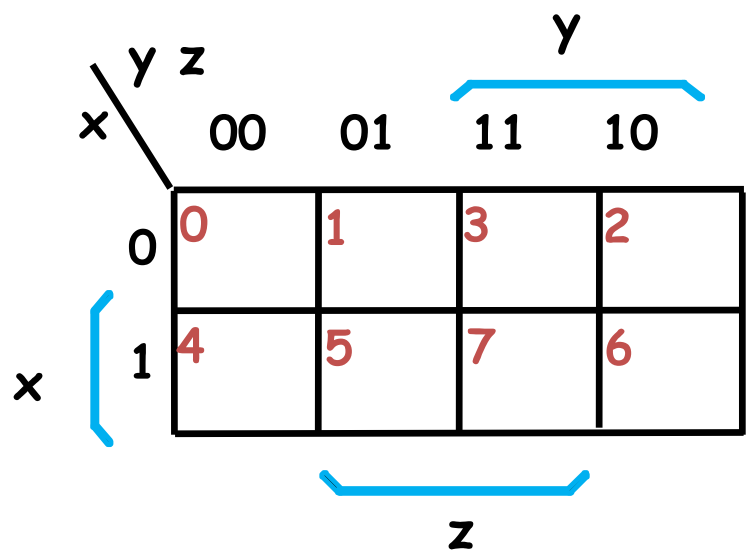
x \ yz	yz=00	yz=01	yz=11	yz=10
x=0	$\overline{x}\,\overline{y}\,\overline{z}$	$\overline{x}\,\overline{y}\,z$	$\overline{x}\,y\,z$	$\overline{x}\,y\,\overline{z}$
x=1	$x\,\overline{y}\,\overline{z}$	$x\,\overline{y}\,z$	$x\,y\,z$	$x\,y\,\overline{z}$

- Note that if the binary value for an index differs in one bit position, the minterms are adjacent on the K-Map

Alternative Map Labeling

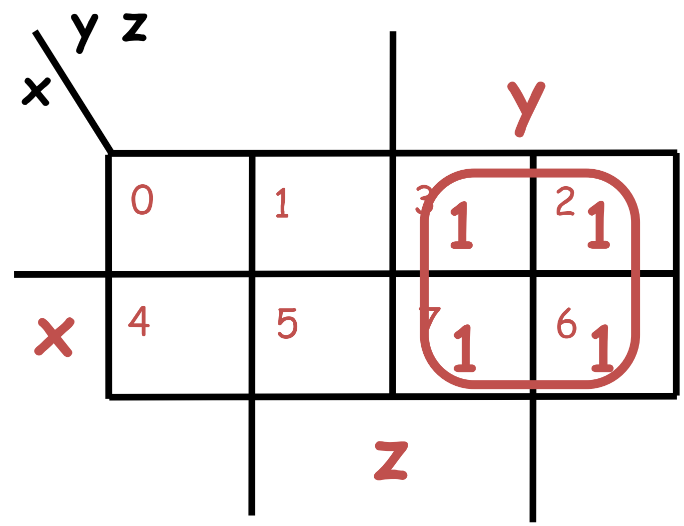
- ✓ Map use largely involves:
  - Entering values into the map, and
  - Reading off product terms from the map.
- ✓ Alternate labelings are useful:





Example: Combining Squares

- ✓ Example: Let
- ✓ $F(x, y, z) = \sum_m (2, 3, 6, 7)$



- ✓ Applying the Minimization Theorem three times:

$$F(x,y,z) = \bar{x}yz + xyz + \bar{x}y\bar{z} + xy\bar{z}$$
$$= yz + y\bar{z}$$
$$= y$$

- ✓ Thus the four terms that form a 2 × 2 square correspond to the term "y".

Combining Squares

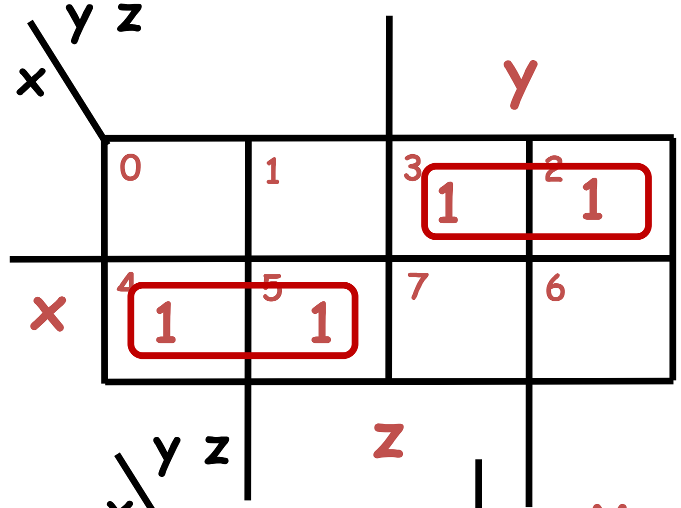
- ✓ By combining squares, we reduce number of literals in a product term, reducing the literal cost, thereby reducing the other two cost criteria
- ✓ On a 3-variable K-Map:
  - One square represents a minterm with three variables
  - Two adjacent squares represent a cube that is product term with two variables
  - Four "adjacent" terms represent a cube that is product term with one variable
  - Eight "adjacent" terms is the function of all ones (no variables) is a tautology $f^1=1$.

Example Functions

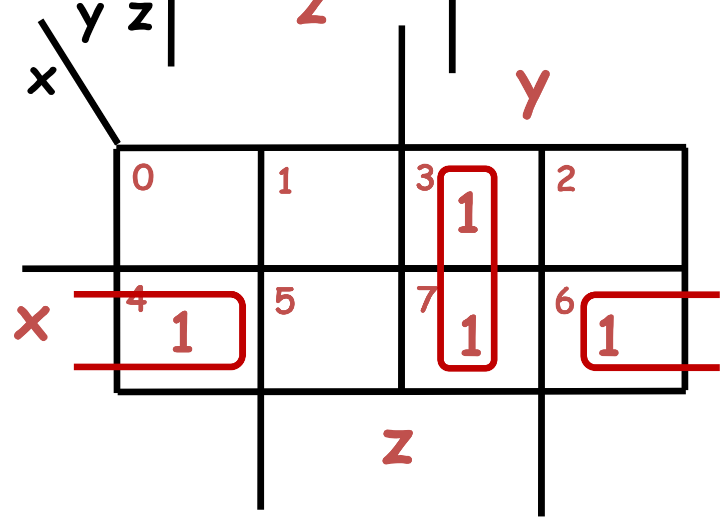
- ✓ By convention, we represent the minterms of F by a "1" in the map and leave the minterms of blank $\overline{F}$
- ✓ Example:

$F(x, y, z) = \sum_m (2, 3, 4, 5)$

$F(x, y, z) = \overline{x}\, y + x\, \overline{y}$



- ✓ Example:
- ✓ $F(x, y, z) = \sum_m (3, 4, 6, 7)$
- ✓ $F(x, y, z) = y\, z + x\, \overline{z}$
- ✓ Learn the locations of the 8 indices based on the variable order shown (x, most significant and z, least significant) on the map boundaries



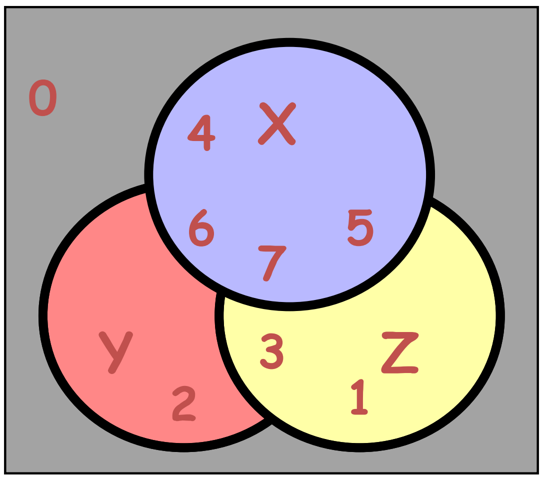
Three-Variable Maps

- ✓ Reduced literal product terms for SOP standard forms correspond to cubes i.e. to rectangles on the K-maps containing cell counts that are powers of 2.
- ✓ Rectangles of 2 cells represent 2 adjacent minterms; of 4 cells represent 4 minterms that form a "pairwise adjacent" ring.

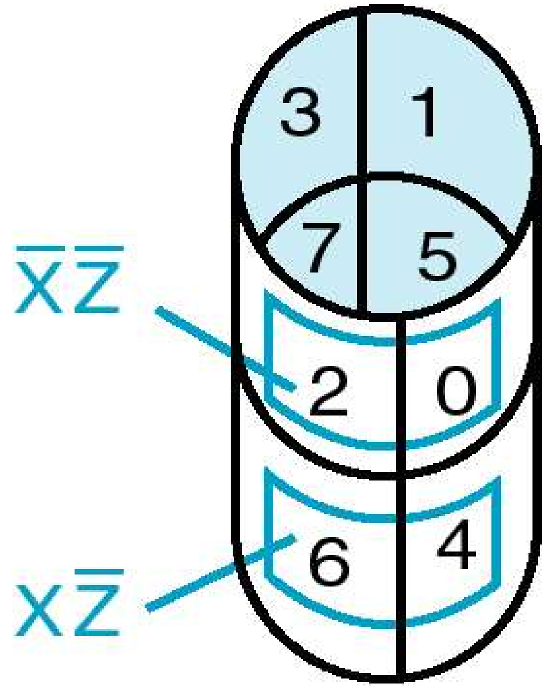
Three-Variable Maps

✓ Topological warps of 3-variable K-maps that show *all* adjacencies:

- Venn Diagram



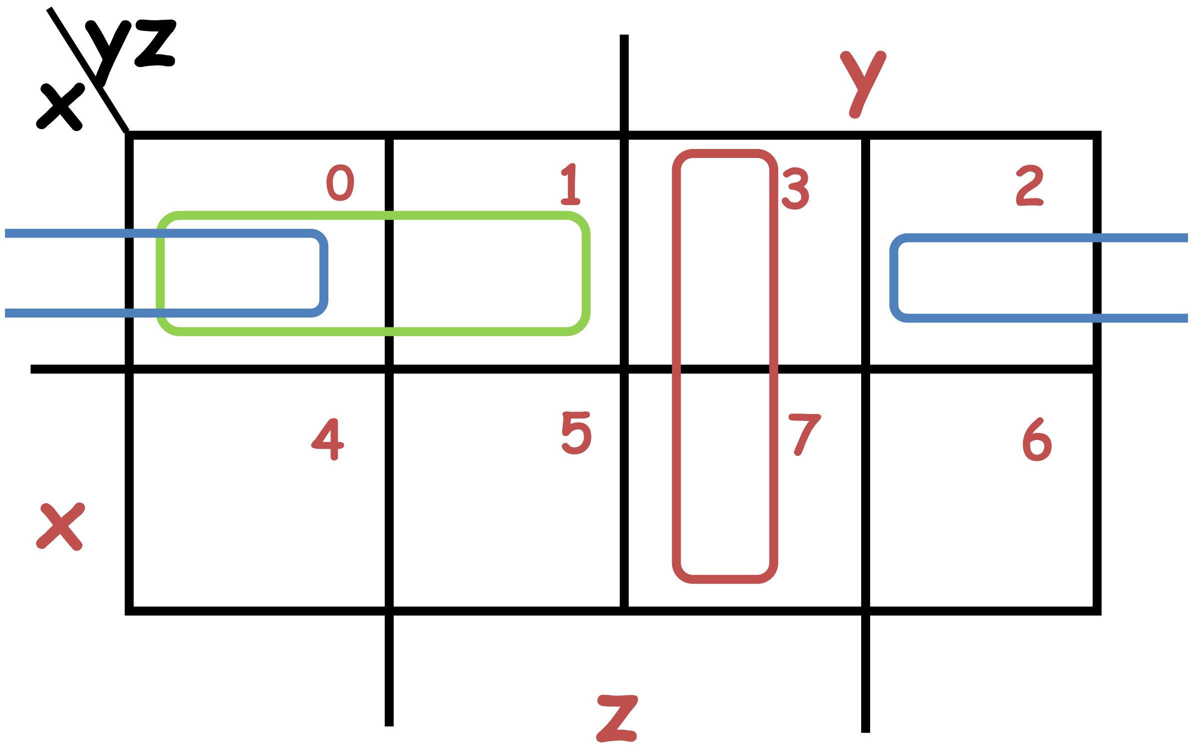
- Cylinder



x \ yz	$\bar{y}\bar{z}$	$\bar{y}z$	yz	$y\bar{z}$
$\bar{x}$	0	1	3	2
x	4	5	7	6

Three-Variable Maps

- ✓ Example Shapes of 2-cell Rectangles:

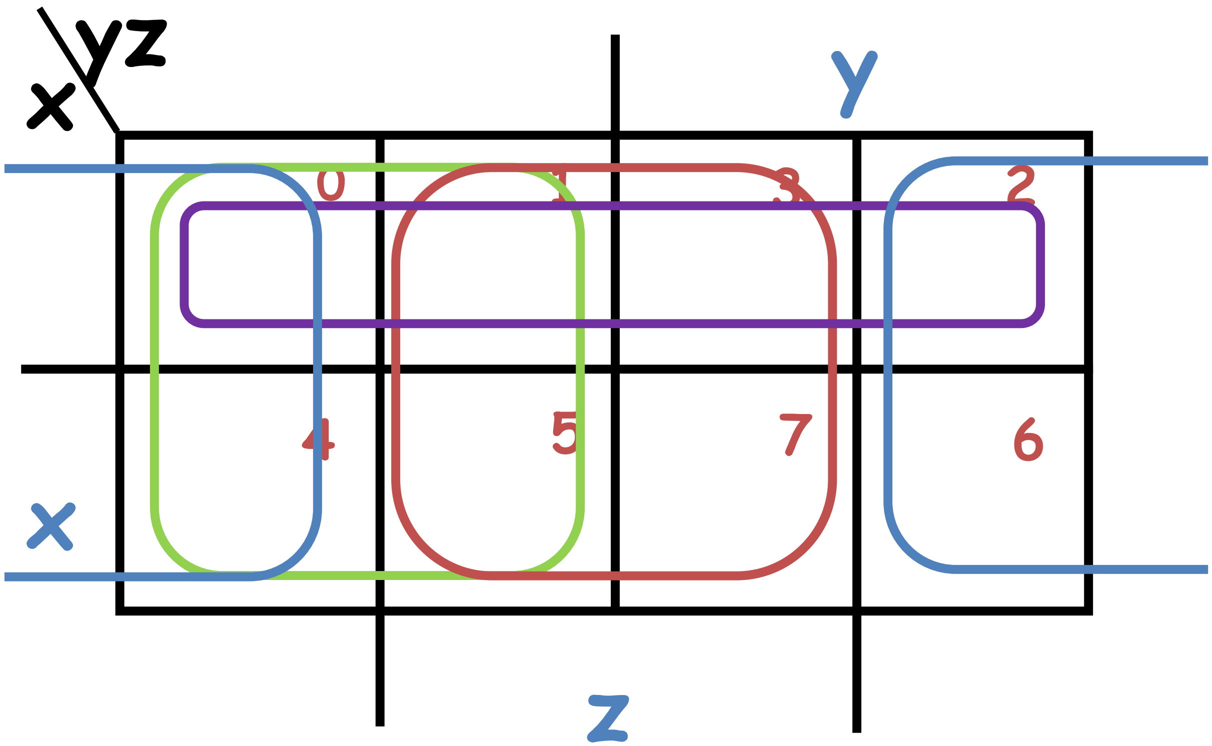


- ✓ Read off the product terms for the rectangles shown

$x'z'$ $x'y'$ yz

Three-Variable Maps

- ✓ Example Shapes of 4-cell Rectangles:

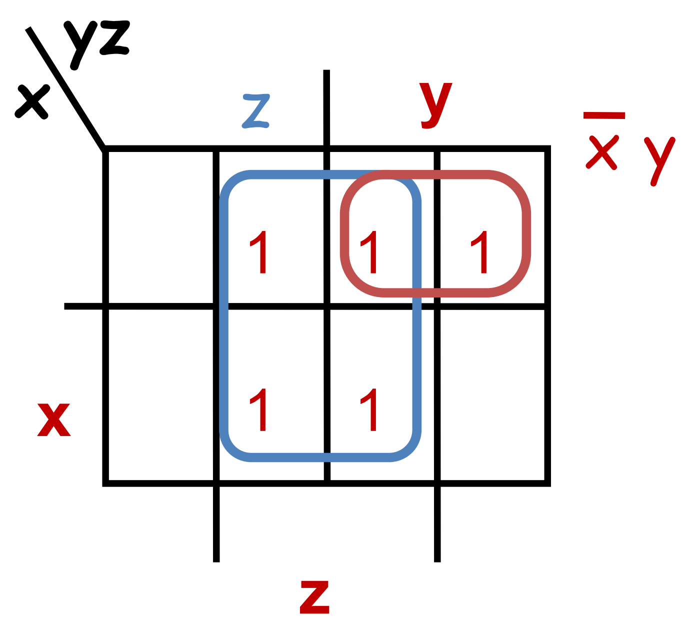


- ✓ Read off the product terms for the rectangles shown

z' z x' y'

Three Variable Maps

- K-Maps can be used to simplify Boolean functions by a systematic methods. Terms are selected to cover the "1s"in the map.
- Example: Simplify $F(x, y, z) = \sum_m (1, 2, 3, 5, 7)$



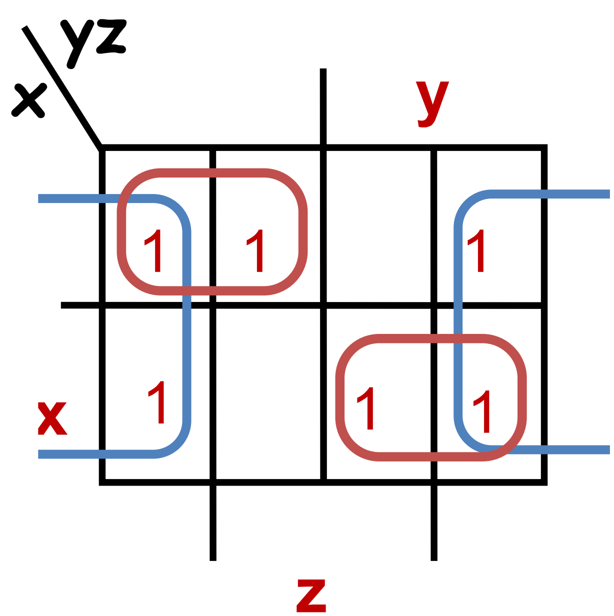
$F(x, y, z) = z + \bar{x}y$

Three-Variable Map Simplification

✓ Use a K-map to find an optimum SOP equation for

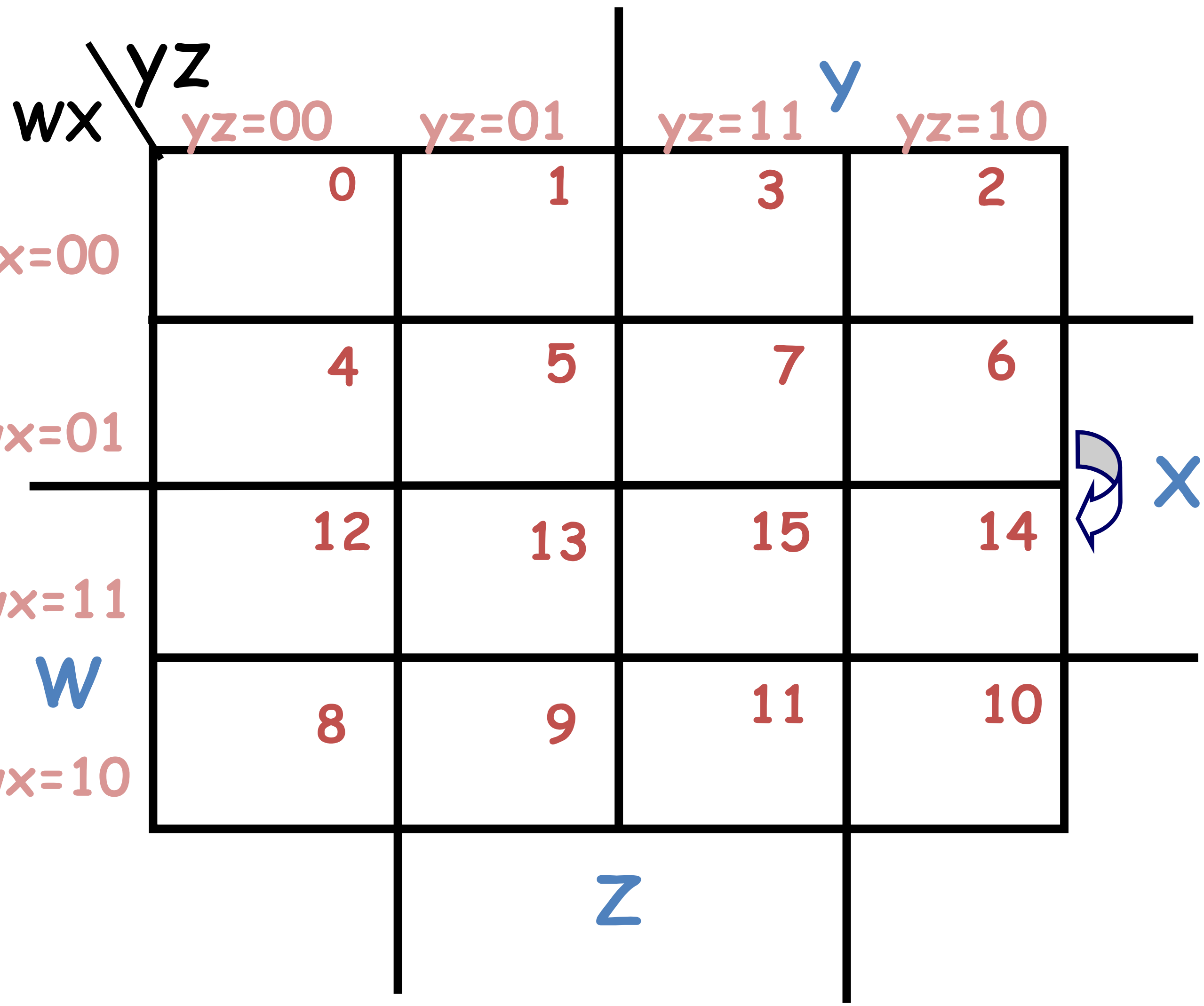
$$F(x, y, z) = \sum_m (0, 1, 2, 4, 6, 7)$$

$$F = \overline{z} + \overline{x}\,\overline{y} + x\,y$$



Four Variable Maps

- ✓ Map and location of minterms:

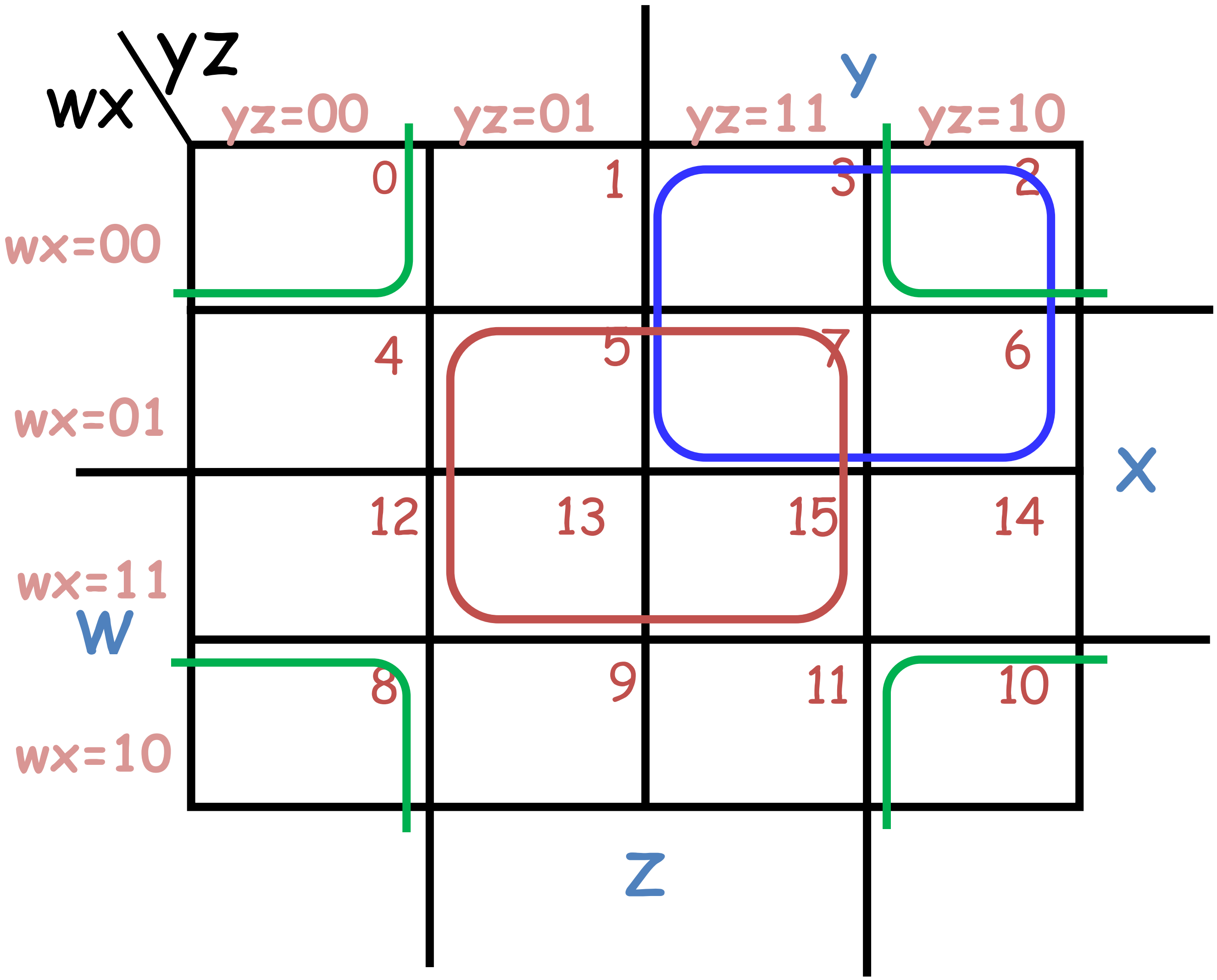


Four Variable Terms

- ✓ Four variable maps can have rectangles corresponding to:
  - A single 1 = 4 variables, (i.e. Minterm)
  - Two 1s = 3 variables,
  - Four 1s = 2 variables
  - Eight 1s = 1 variable,
  - Sixteen 1s = zero variables (i.e. Constant "1")

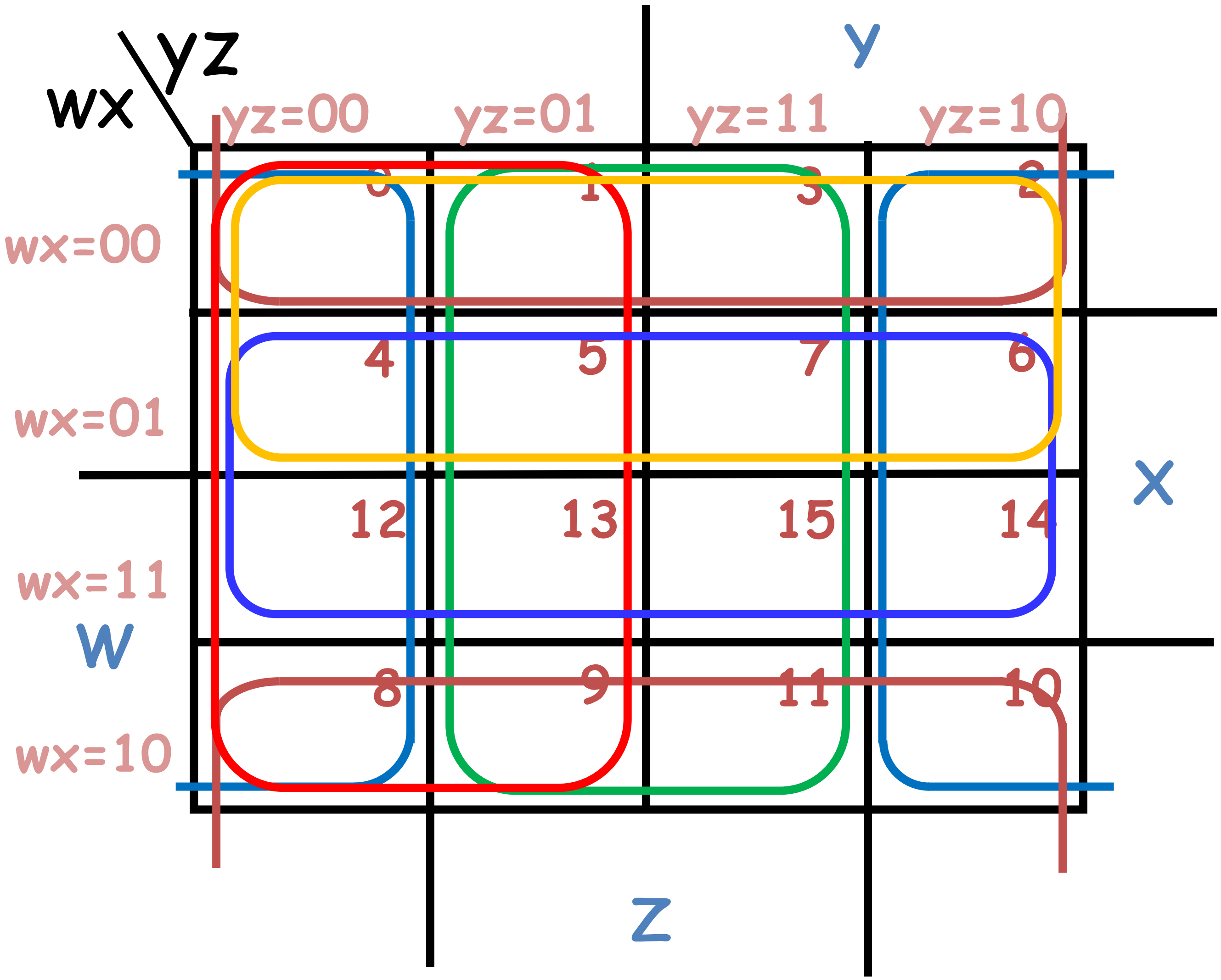
Four-Variable Maps

✓ Example Shapes of Rectangles:



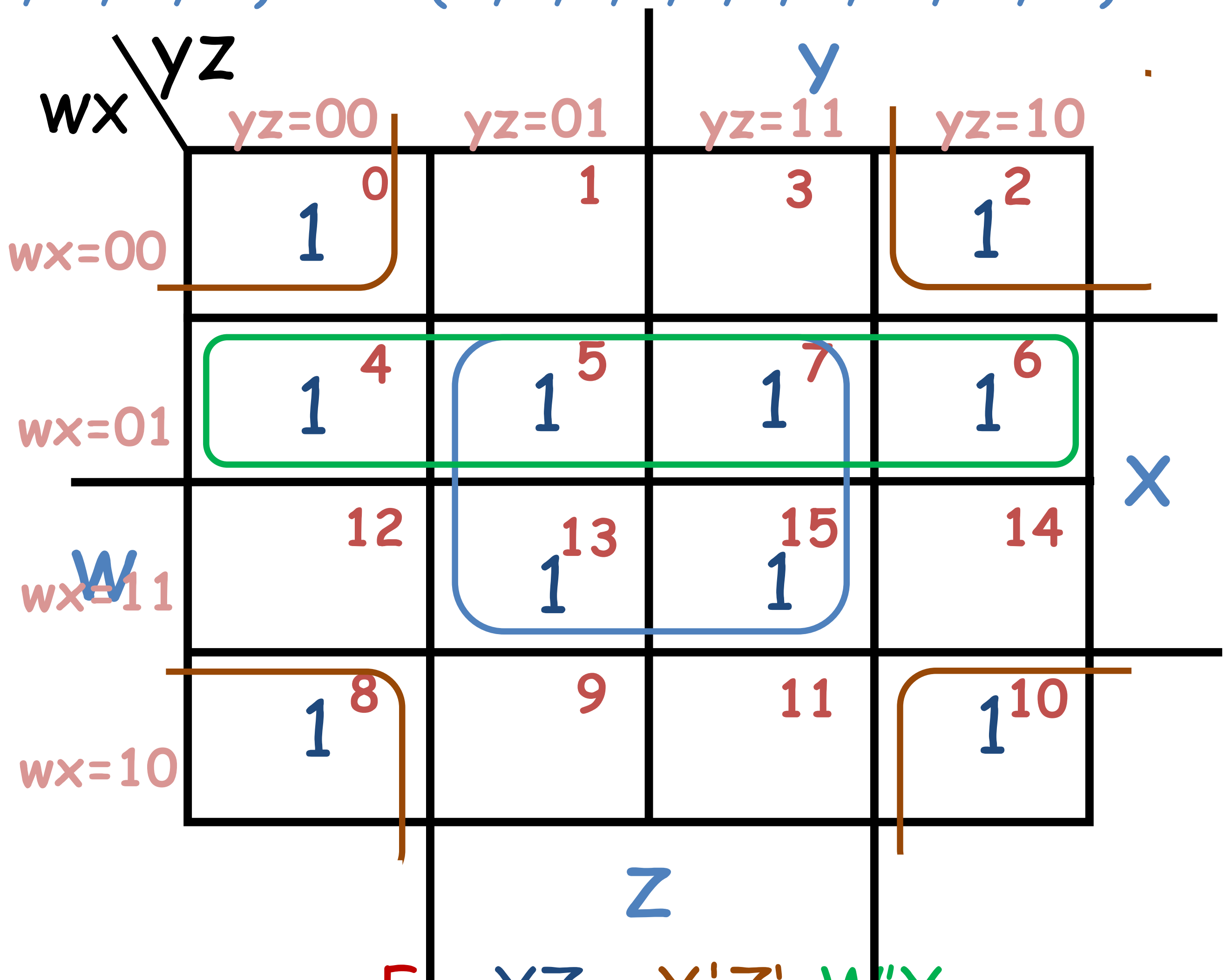
Four-Variable Maps

✓ Example Shapes of Rectangles:



Four-Variable Map Simplification

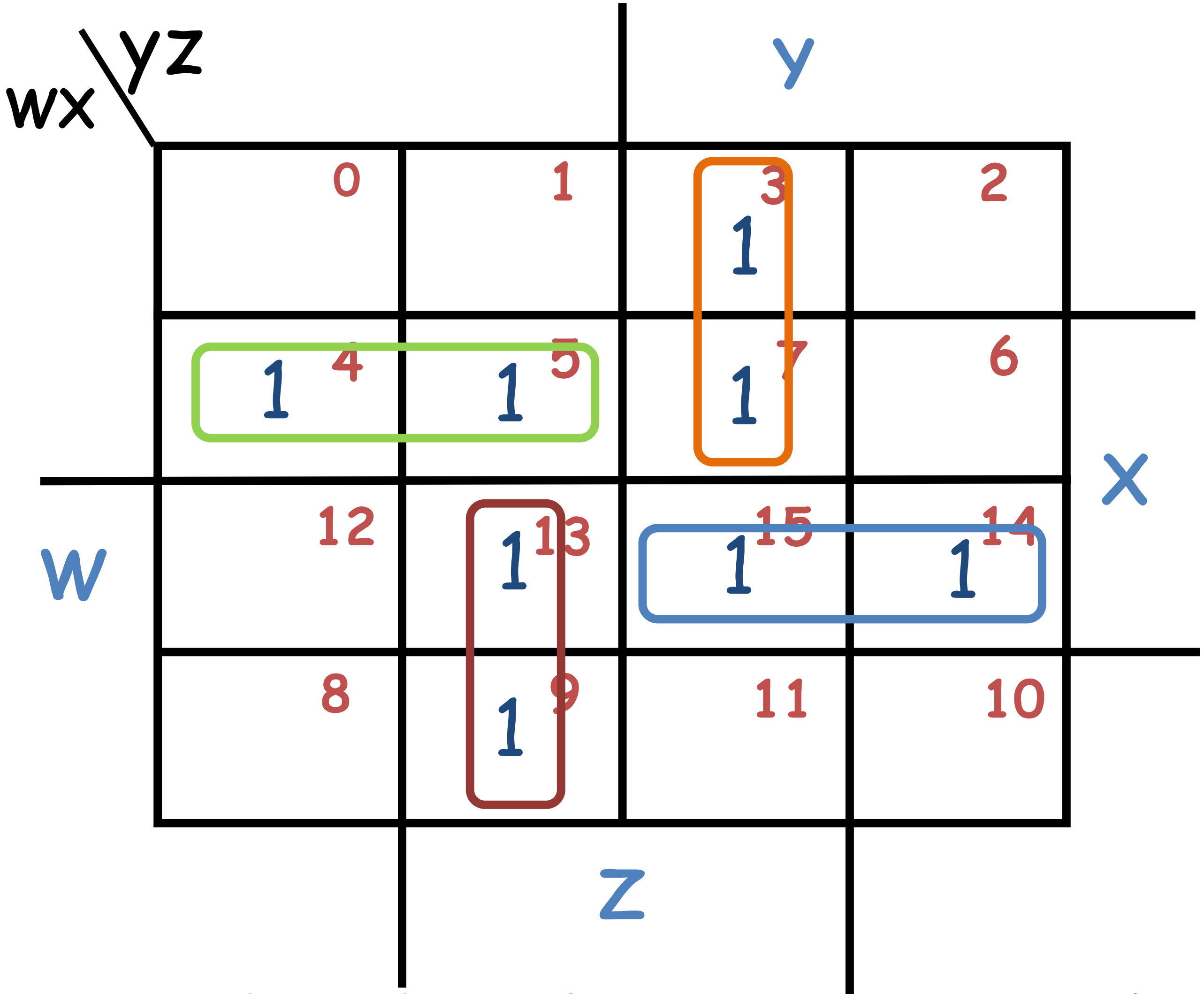
- $F(W,X,Y,Z) = \sum m(0,2,4,5,6,7,8,10,13,15)$



F = XZ + X'Z' + W'X

Four-Variable Map Simplification

$F(W,X,Y,Z) = \sum m(3,4,5,7,9,13,14,15)$



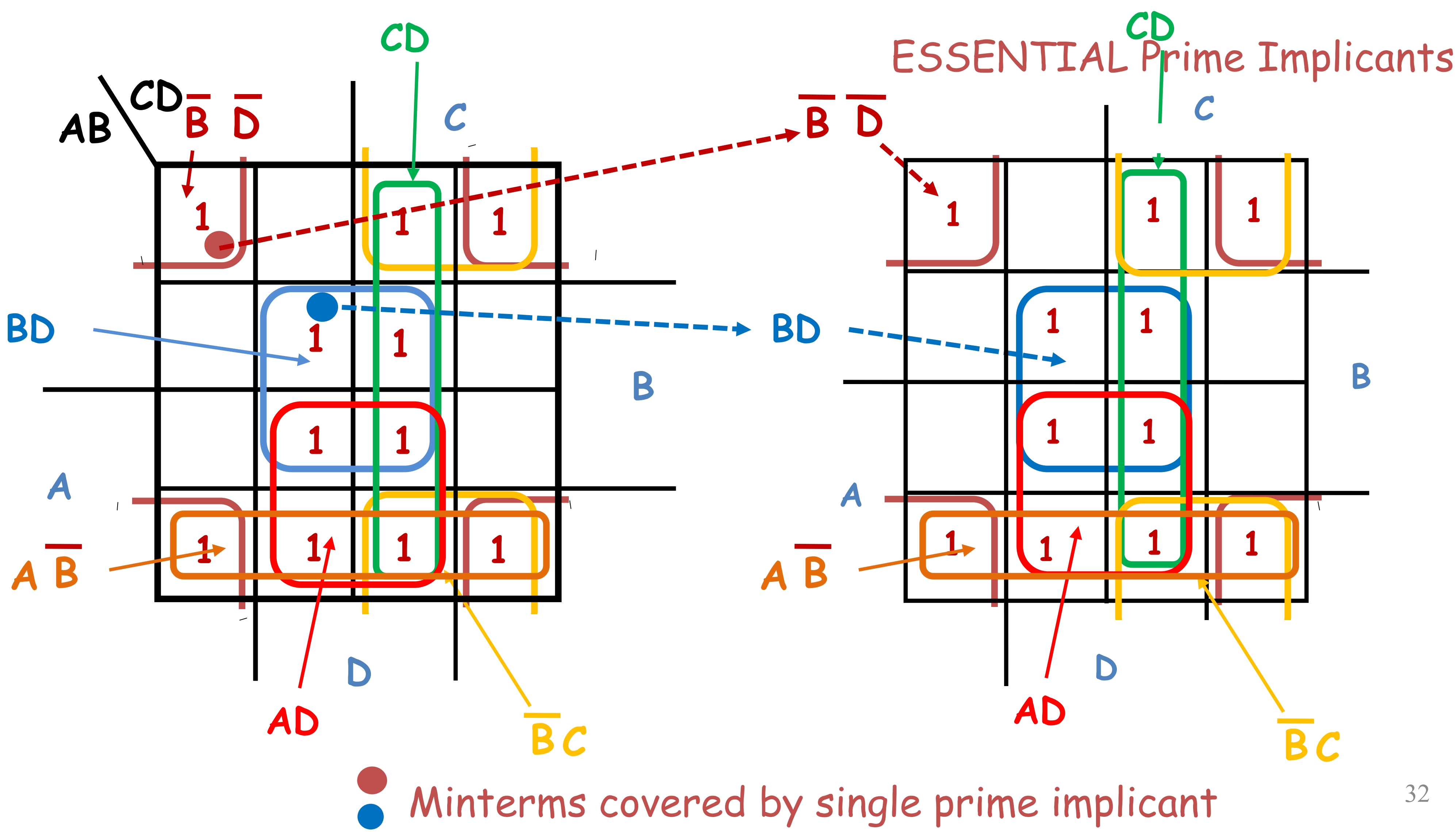
$F = W'XY' + W'YZ + WXY + WY'Z$

Systematic Simplification

- ✓ A Prime Implicant is a cube i.e. a product term obtained by combining the maximum possible number of adjacent squares in the map into a rectangle with the number of squares a power of 2.
- ✓ A prime implicant is called an Essential Prime Implicant if it is the only prime implicant that covers (includes) one or more minterms.
- ✓ Prime Implicants and Essential Prime Implicants can be determined by inspection of a K-Map.
- ✓ A set of prime implicants "*covers all minterms*" if, for each minterm of the function, at least one prime implicant in the set of prime implicants includes the minterm.

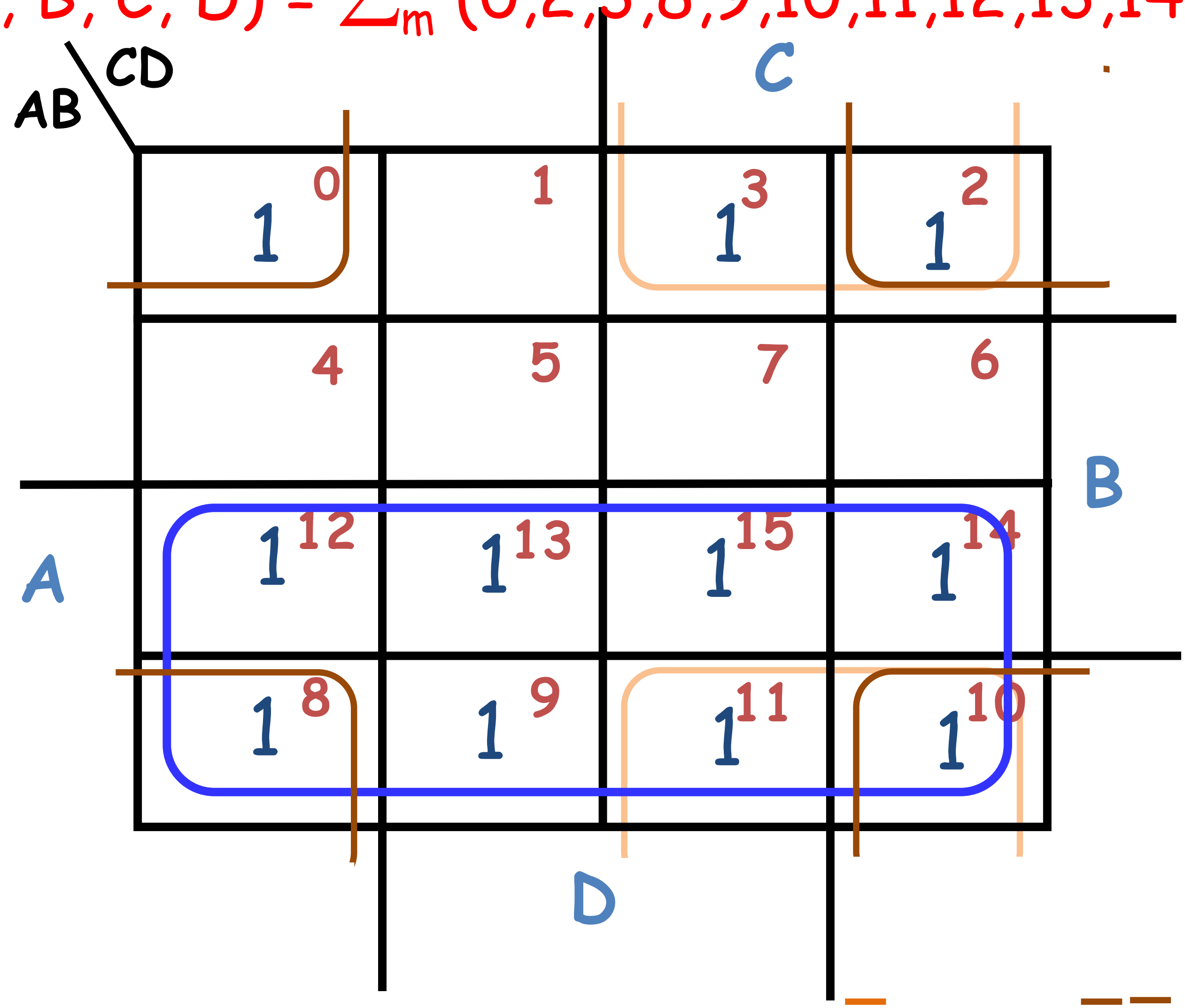
Example of Prime

✓ Find ALL Prime Implicants



Prime Implicant Practice

- ✓ Find all prime implicants for:
- ✓ $F(A, B, C, D) = \sum_m (0,2,3,8,9,10,11,12,13,14,15)$

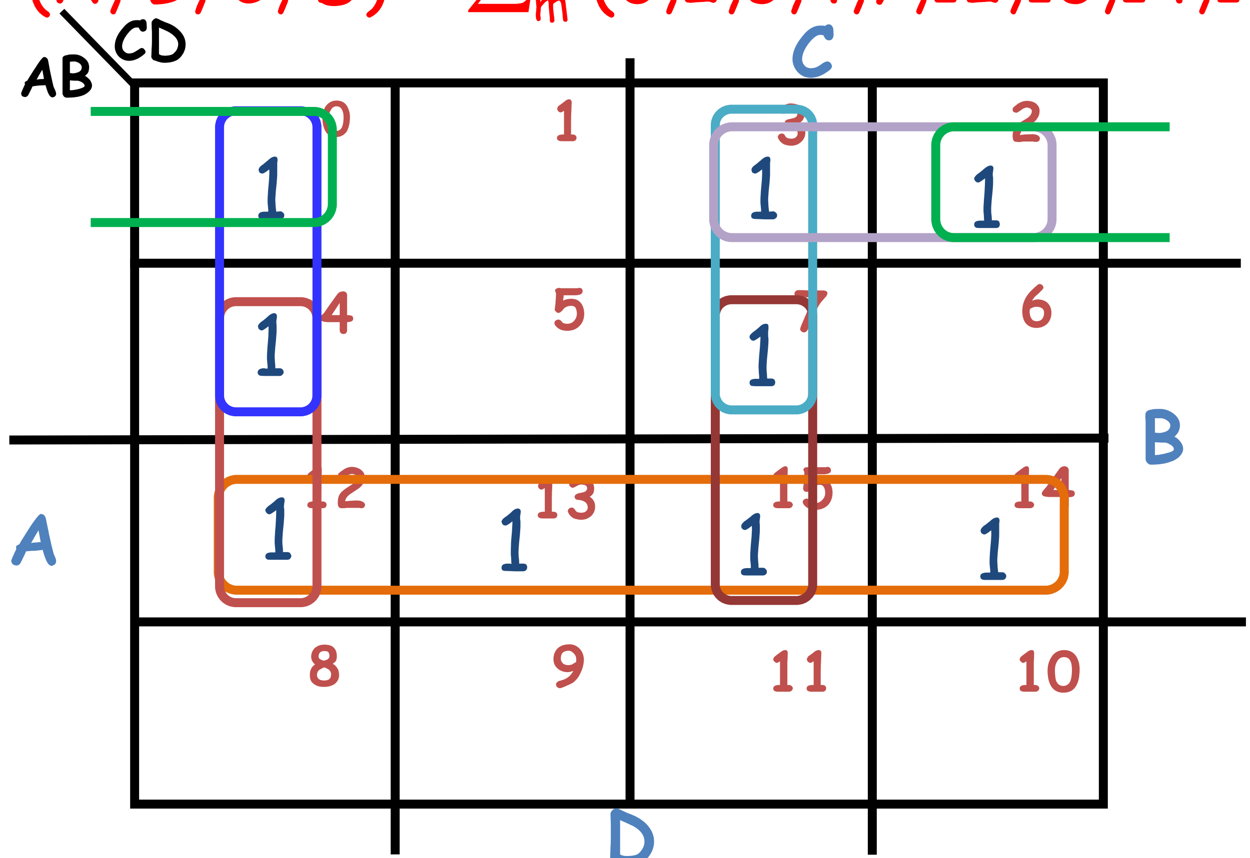


Prime implicants are: A, $\overline{B}C$, and $\overline{B}\,\overline{D}$

Another Example

✓ Find all prime implicants for:

$$F(A,B,C,D) = \sum_m (0,2,3,4,7,12,13,14,15)$$



AB, $B\bar{C}\bar{D}$, $\bar{A}\bar{C}\bar{D}$, $\bar{A}\bar{B}\bar{D}$, $\bar{A}\bar{B}C$, $\bar{A}CD$, BCD

$$\sum_m = AB + \bar{A}\bar{C}\bar{D} + \bar{A}CD + \bar{A}\bar{B}C \quad \sum_c = \sum_m + \bar{A}\bar{B}\bar{D} + BCD + B\bar{C}\bar{D}$$